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EQUILIBRIUM WITH PUBLIC GOODS

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Abstract

This paper investigates the provision of public goods in urban areas. Equilibrium conditions for public good and housing consumption are developed for regimes of property taxes and head taxes. Voter unanimity characterizes the property tax solution, while the median voter principle is operative in the head tax solution.

A Note on Urban Spatial
Equilibrium with Public Goods

The purpose of this note is to derive equilibrium conditions for an urban area with a pure public good, where the level of the good is decided upon by majority voting. The available papers dealing with spatial equilibrium with public goods are unsatisfactory for a variety of reasons. Fisch [3] analyzes a model where the public good has limited spatial scope, but he never explains how the public good output is divided up across neighborhoods. Wile [4] discusses a model with an extremely restrictive property tax formulation. Barr's analysis [1] is closest to being satisfactory, but it suffers from lack of generality due to the use of Cobb-Douglas functions.

The models considered in this note are built on the assumption that people consume only a pure public good and housing. It will be seen that this assumption implies that housing consumption is constant over space, an unrealistic conclusion. However, since the intent of the discussion is to provide insight into spatial equilibrium with a pure public good without being completely general or realistic, the restrictive consumption requirement is defensible. The first model below has property taxation, while in the second model, revenue is raised through head taxes.

As usual, all residents in the city commute to the CBD where they engage in a production process. The residents consume housing, h, and a pure public good, z.[1] It is assumed that all residents know the government's cost function for the public good, $C(z)$. Each consumer earns income y, which is exogenous,[2] and incurs commuting cost $t(x)$, where x is the radial distance from his residence to the CBD.

Since z is a pure public good, the level of consumption does not depend on distance, x. Because the utility level of consumers must be invariant across space, this implies that h also does not depend on x; housing consumption does not vary over space as in the standard urban model. The property tax rate Θ which balances the city budget is $\Theta = C(z)/V$, where V is the total value of housing in the urban area. From the budget constraint $ph(1 + \Theta) = y - t(x)$, we have

$$p = \frac{y - t(x)}{h(1 + C(z)/V)} \equiv p(x, V, z, h), \tag{1}$$

where p is the unit price of housing. It is clear that the function p helps determine V, and that the p function generated by a particular value of V may not itself imply the original V value. We will see below that in equilibrium this must happen; (1) will be an ingredient in the determination of equilibrium.

Perfectly competitive housing producers maximize profits, $pH(K, \ell) - qK - r\ell$, where q, the price of capital, K, is exogenous and r, the rental price of land, ℓ, is endogenous and free to vary with x. The producer first-order and zero-profit conditions

$$\begin{aligned} p(x, V, z, h)H_1(K, \ell) &= q \\ p(x, V, z, h)H_2(K, \ell) &= r \\ p(x, V, z, h)H(K, \ell) - qK - r\ell &= 0 \end{aligned} \tag{2}$$

yield $r = r(x, V, z, h)$, the land rent function which makes maximized profit equal to zero at all x. Housing output per unit of land, a quantity which also emerges from (2), is $\phi(x, V, z, h)$.

In deciding what level of the public good to vote for, consumers maximize utility subject to their budget constraints. Substituting for h in u, the maximand becomes

$$u\left(\frac{y - t(x)}{p(1 + C(z)/V)} , z \right). \tag{3}$$

The function p has already been chosen to make utility constant across x for any given (h, z) pair; consumer maximization decides which pair is chosen. Since consumers are perfectly competitive, they take p as given and they assume their decision has no effect on V, the total value of property in the city. The first-order condition is

$$\frac{u_2}{u_1} = \frac{(y - t(x))pC'(z)/V}{p^2(1 + C(z)/V)^2} = \frac{hC'(z)/V}{(1 + C(z)/V)} . \tag{4}$$

For fixed V and u, equation (4) implies that the chosen (h, z) bundle does not depend on x; people at different locations are unanimous in their choice of the level of the public good. We will see below that this is not true when head taxes are levied.

The equilibrium conditions for the urban area are as follows:

$$r(\bar{x}, V, z, h) = r_A \tag{5}$$

$$\int_0^{\bar{x}} (\alpha(x)x\phi(x, V, z, h)/h)dx = n \tag{6}$$

$$\int_0^{\bar{x}} \alpha(x)xp(x, V, z, h)\phi(x, V, z, h)dx = V \tag{7}$$

$$\frac{u_2(h, z)}{u_1(h, z)} = \frac{hC'(z)/V}{(1 + C(z)/V)} \tag{8}$$

$$u = u(n, z) \tag{9}$$

Equation (5) says urban land rent falls to r_A, the agricultural land rent, at $\bar{x}$, the urban periphery. Equation (6) says that the urban area houses its population, n. In the integrand, $\alpha(x)$ is the number of radians of land at x available for housing, $\alpha(x)xdx$ is land area at x available for housing, and ϕ/h is households per acre, or population density. Equation (7) says that the aggregate property value underlying the p and ϕ function is indeed the value generated by the system. Equation (8) is the consumer equilibrium condition, and (9) gives the utility level. The variables are h, z, V, $\bar{x}$, n, and u, but there are only five equations; either u or n must be exogenous. Letting n be exogenous means the city is closed to migration and the utility level is determined internally; u exogenous means n adjusts until the utility level equals the prevailing level in the outside world.

The nature of this equilibrium is noteworthy; people are perfectly competitive with respect to V, believing their decision has no effect on aggregate property value. In equilibrium, however, this myopia must be validated; the fixed V value which enters into everyone's decision-making must be the one generated by the system. The other interesting property of the model is voter unanimity: for fixed V and u, everyone agrees on the levels of h and z. In particular, for the equilibrium values of V and u, everyone desires to consume the equilibrium levels of h and z. In the model with head taxes which is developed next, there is disagreement about the correct level of the public good even in equilibrium; the majority voting process is central to the outcome.

With property taxes, the tax payment in equilibrium is $\Theta ph = \Theta(y - t(x))/(1 + \Theta)$, where Θ is the equilibrium property tax rate. This quantity decreases with x. With head taxes, however, each individual pays a tax equal to $C(z)/n$.

In what follows it will be convenient to assume that $C(z) \equiv cz$. The budget constraint is $ph + cz/n = y - t(x)$, which yields

$$p = \frac{y - t(x) - cz/n}{h} \equiv p(x, z, h, n) \tag{10}$$

As in (2), the p function generates land rent, $r(x, z, h, n)$, and housing output per unit land, $\phi(x, z, h, n)$. The maximand for the consumer problem is

$$u\left(\frac{y - t(x) - cz/n}{p}, z\right), \tag{11}$$

and the first-order condition is

$$\frac{u_2(h, z)}{u_1(h, z)} = \frac{c}{np(x, z, h, n)} \tag{12}$$

For the moment, assume u and n are given. Fix x at x' and choose the z', h' pair which satisfies (12) and generates the given utility level. This pair is the one at which the budget line for an individual living at x' is tangent to the indifference curve with the given utility level. The choice of z and h for given n determines the housing price p from (10) at each x, and hence determines the budget lines for residents at all x values. These budget lines have the equation $p(x, z', h', n)h + cz/n = y - t(x)$, with slope $-c/np(x, z', h', n)$. Each line passes through the point (h', z') because of the definition of p in (10). But since $\partial p/\partial x < 0$ from (10),

$$\frac{u_2(h', z')}{u_1(h', z')} \gtrless \frac{c}{np(x, z', h', n)} \quad \text{as } x \lessgtr x'. \tag{13}$$

This means that people consuming (h', z') at x < x' have an MRS which exceeds the slope of their budget line and people consuming (h', z') at x > x' have an MRS which is less than the slope of their budget line. The only distance at which people satisfy their first-order conditions for the pair (h', z') is x', and this was true by construction (see Figure 1).

We may ask whether a given (h, z) pair will be supportable as a majority voting equilibrium. Suppose someone in the urban area, as above, satisfies his first-order condition, and the resulting (h, z) pair is the consumption bundle for the residents of the urban area. If we require that this person lives at x', the equilibrium conditions for the urban area are

$$r(\bar{x}, z, h, n) = r_A \tag{14}$$

$$\int_0^{\bar{x}} (\alpha(x)x\phi(x, z, h, n)/h)dx = n \tag{15}$$

$$\frac{u_2(h, z)}{u_1(h, z)} = \frac{c}{np(x', z, h, n)} \quad . \tag{16}$$

$$u = u(h, z) \tag{17}$$

The variables are h, z, u, n, and $\bar{x}$, and as there are only four equations, either u or n must be exogenous. Let h', z', and u' be the h, z, and u values from the solution to (14-17). We will now show that this solution cannot be a majority voting equilibrium unless x' is the median distance from the CBD in the urban area. Suppose x' is beyond the median distance, that is, the population living between the CBD and x' exceeds that living between x' and $\bar{x}$. In Figure 2, suppose a candidate running for office proposed a public good level of z". Clearly, any person with a budget line such as B will be better off consuming at b than at a, and hence he will vote for this candidate. The person with budget line A will be indifferent between consuming at a and voting for the candidate so as to consume at e. All people with budget lines with slopes less negative than that of A, namely all people between some $x < x'$ and $x = 0$, will vote for z". Since x' is greater than the median x, we can always find a point like e and its associated budget line such that the number of people having budget lines with less negative slopes is greater than n/2. The z coordinate for such a point will always win over the z coordinate of point a under

$$r(\overline{x}, z, h, n) = r_A \tag{14}$$

$$\int_0^{\overline{x}} (\alpha(x)x\phi(x, z, h, n)/h)dx = n \tag{15}$$

$$\frac{u_2(h, z)}{u_1(h, z)} = \frac{c}{np(x', z, h, n)} \quad . \tag{16}$$

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The variables are h, z, u, n, and $\overline{x}$, and as there are only four equations, either u or n must be exogenous. Let h', z', and u' be the h, z, and u values from the solution to (14-17). We will now show that this solution cannot be a majority voting equilibrium unless x' is the median distance from the CBD in the urban area. Suppose x' is beyond the median distance, that is, the population living between the CBD and x' exceeds that living between x' and $\overline{x}$. In Figure 2, suppose a candidate running for office proposed a public good level of z". Clearly, any person with a budget line such as B will be better off consuming at b than at a, and hence he will vote for this candidate. The person with budget line A will be indifferent between consuming at a and voting for the candidate so as to consume at e. All people with budget lines with slopes less negative than that of A, namely all people between some x < x' and x = 0, will vote for z". Since x' is greater than the median x, we can always find a point like e and its associated budget line such that the number of people having budget lines with less negative slopes is greater than n/2. The z coordinate for such a point will always win over the z coordinate of point a under

majority voting. The same argument holds if x' is less than the median x. Only if x' equals the median x will the associated consumption bundle be undefeatable under majority voting. Any other z value will always draw the votes of less than half of the population. Thus to make (14-17) into part of a true equilibrium system, we must add the condition

$$\int_0^{x'} (\alpha(x)x\phi(x,\ z,\ h,\ n)/h)dx = n/2, \tag{18}$$

which says that x' is the median distance from the CBD.

The structure of equilibrium with head taxes is more complex and interesting than the structure of the property tax equilibrium. What is new about this analysis is the emergence in a spatial context of the median voter principle, which is familiar in non-spatial models of public goods provision.

If it assumed that land instead of housing is consumed by urban residents, it is easy to investigate the disparity between the utility level generated by majority voting and the maximum level attainable by urban residents. Denoting land by ℓ and land rent by r, we must have

$$r = \frac{y - t(x) - cz/n}{\ell} \equiv r(x,\ z,\ \ell,\ n) \tag{19}$$

The equilibrium conditions for the urban area analogous to (14) - (18) are, assuming $\alpha(x) = 2\pi$ for simplicity,

$$r(\overline{x},\ z,\ \ell,\ n) = r_A \tag{20}$$

$$\pi\bar{x}^2/\ell = n \tag{21}$$

$$\frac{u_2(\ell, z)}{u_1(\ell, z)} = \frac{c}{nr(x', z, \ell, n)} \tag{22}$$

$$u = u(\ell, z) \tag{23}$$

$$\pi(x')^2/\ell = n/2 \tag{24}$$

In the closed city case, where n is fixed and u is endogenous, the maximum achievable utility level is given by the solution to the problem: maximize $u(\ell, z)$ subject to $\pi\bar{x}^2/\ell = n$ and $y - t(\bar{x}) - cz/n = r_A\ell$. The equations characterizing the solution are (20), (21), (23) and

$$\frac{u_2(\ell, z)}{u_1(\ell, z)} = \frac{c}{n(r_A + \frac{t'(\bar{x})\bar{x}}{2\ell})} \tag{25}$$

Since (25) is different from (23), the majority voting equilibrium does not maximize u. It is easy to show that if all city residents recognize the dependence of r on ℓ, then there will be unanimity in the voting process, and the outcome will maximize urban utility. From (19), (20), and (21)

$$r = \frac{t(\sqrt{n\ell/\pi}) - t(x)}{\ell} + r_A, \tag{26}$$

where $\sqrt{n\ell/\pi} = \bar{x}$. Consumer utility maximization subject to the budget constraint with r expressed as in (26) generates condition (25). The urban

equilibrium is then characterized by (20), (21), (25), and (23), which yield the maximal value of u. This result is not surprising; it says that if consumers engage in non-competitive behavior, they can raise their level of utility. Wile has noticed a similar fact in his analysis, but he attributes the result to the consumers' failure to account for the spatial externality they impose on others: higher land consumption by one individual imposes higher transportation costs on others due to the expansion of the city. A better explanation is more down-to-earth: people can reach a higher level of utility when they recognize the influence of their consumption on the prices they pay than when they ignore this effect.

It should not be inferred from the disparity between the majority voting utility level and the maximal level that the majority voting solution is Pareto-inefficient. The reason for this is that we have ignored the welfare of the absentee landlords who receive the urban land rent. Certainly, it can be shown that these individuals are made worse off as aggregate urban rent shrinks in moving from the majority voting to the urban utility-maximizing solution.

FOOTNOTES

[1]The analysis could be carried out with a congested instead of a pure public good. That is, z, per capita consumption of the public good could equal $f(x, n)$, where x is output, n is population, and $f_2 < 0$. With a pure public good, $z \equiv x$.

[2]A complete model of an urban area (see, for example, Brueckner [2]), must have endogenous wages and trade-balance conditions. The wage must be partly determined by the labor demand of perfectly competitive CBD producers. Other conditions which are required for equilibrium state that the value of CBD production equals the value of consumption of the good internally plus the value of imports to the urban area of other goods such as capital inputs for housing production. The particular form of the trade balance equations is fairly arbitrary depending on the identity of the good produced in the CBD. Alternatively, a city which is closed to trade must have all prices endogenous. The model we analyse in this note is standard in that the wage is fixed and the CBD production activity is ignored. This is equivalent to assuming constant returns to scale in CBD production so that the wage is independent of output and ignoring trade balance requirements.

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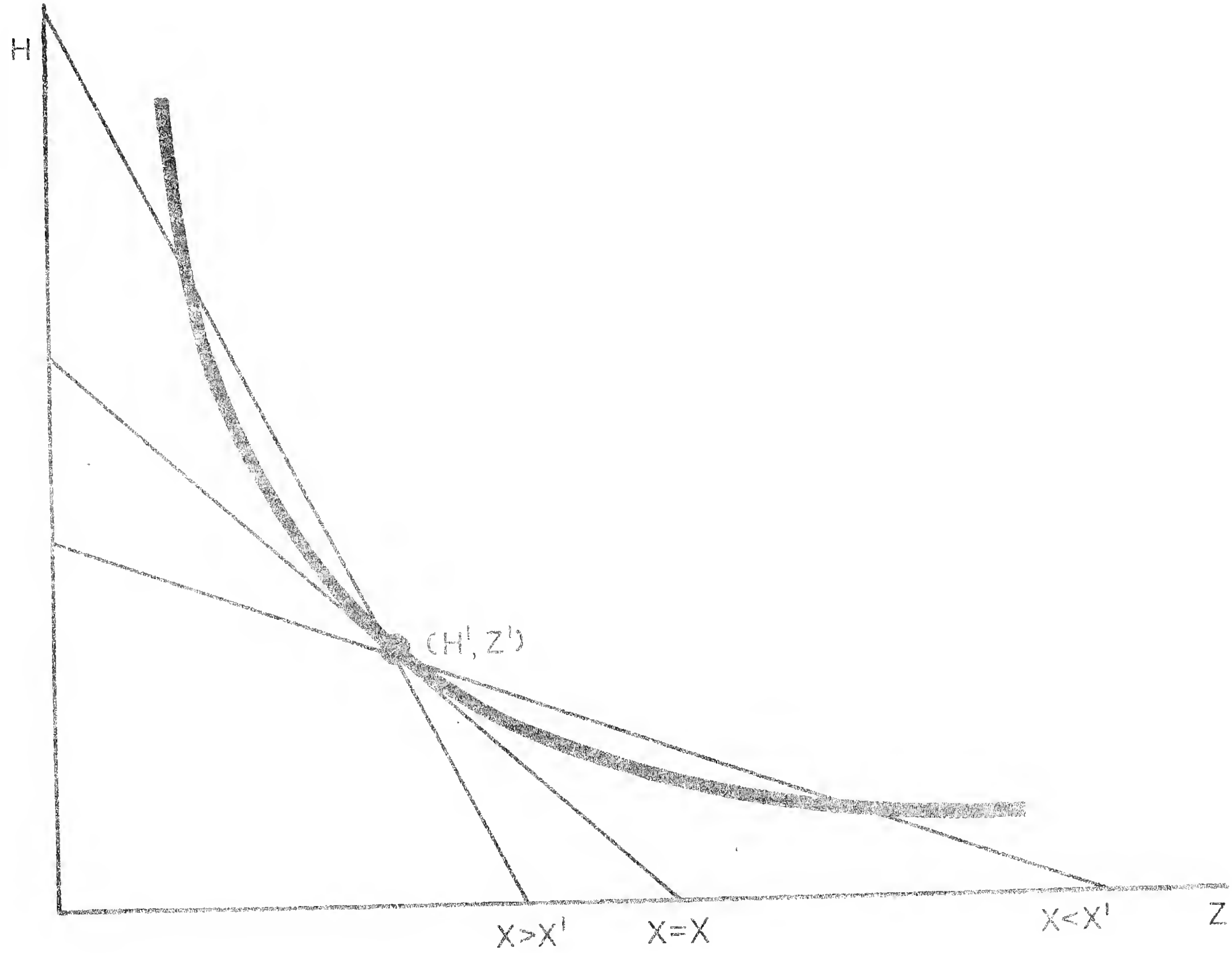

Figure 1.

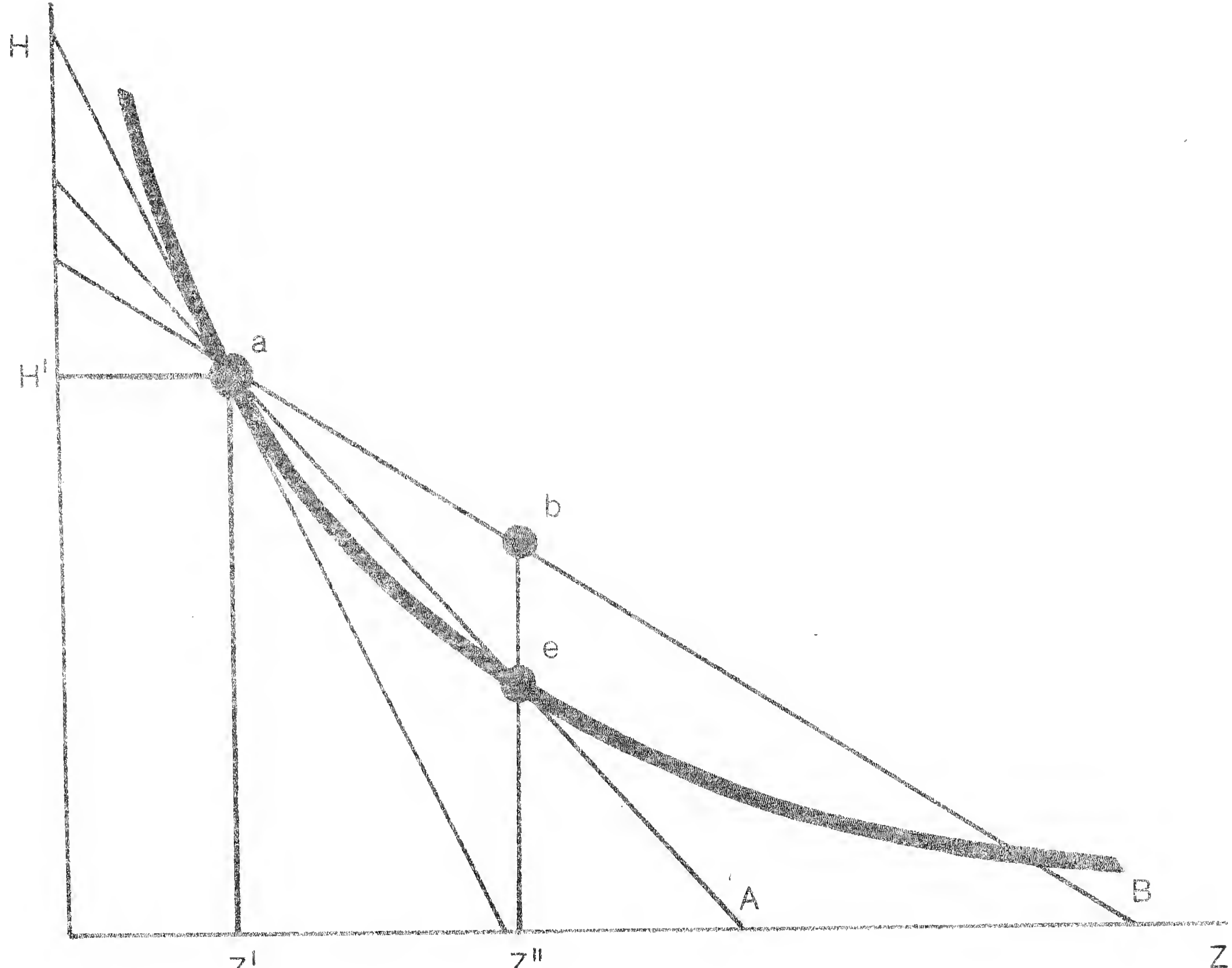

Figure 2.